PP36675. Proposed by Mihaly Bencze.

In any triangle ABC holds:

1) $\sum\left(\dfrac{a^2+16a+80}{16(a+4)}+\dfrac{2}{\sqrt{2(a^2+16)}}\right)r_b r_c \geq \dfrac{3s^2}{2}$;

2) $\sum\left(\dfrac{r_a^2+16r_a+80}{16(r_a+4)}+\dfrac{2}{\sqrt{2(r_a^2+16)}}\right)bc \geq \dfrac{3(s^2+r^2+4Rr)}{2}$.

Solution by Arkady Alt, San Jose, California, USA.

First we will prove that for any $x>0$ holds inequality

(**1**) $\dfrac{x^2+16x+80}{16(x+4)}+\dfrac{2}{\sqrt{2(x^2+16)}} \geq \dfrac{3}{2}$.

Indeed, since $\sqrt{2(x^2+16)} \geq x+4 \Leftrightarrow (x-4)^2 \geq 0$ then

$$\frac{x^2+16x+80}{16(x+4)} = \frac{x^2+16}{16(x+4)}+1 \geq \frac{x^2+16}{16\sqrt{2(x^2+16)}}+1 = \frac{\sqrt{x^2+16}}{16\sqrt{2}}+1$$

and, therefore, by AM-GM Inequality $\dfrac{x^2+16x+80}{16(x+4)}+\dfrac{2}{\sqrt{2(x^2+16)}} =$

$$\frac{\sqrt{x^2+16}}{16\sqrt{2}}+\frac{\sqrt{2}}{\sqrt{x^2+16}}+1 \geq 2\sqrt{\frac{\sqrt{x^2+16}}{16\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{x^2+16}}}+1 = 2\sqrt{\frac{1}{16}}+1 = \frac{3}{2}.$$

1. Applying inequality (**1**) to $x=r_a$ and, cyclic, we obtain

$$\sum\left(\frac{a^2+16a+80}{16(a+4)}+\frac{2}{\sqrt{2(a^2+16)}}\right)r_b r_c \geq \sum\frac{3}{2}r_b r_c.$$

Since $r_a(s-a)=rs \Leftrightarrow r_a=\dfrac{rs}{s-a}$ and $r^2s=(s-a)(s-b)(s-c)$ then

$$\sum r_b r_c = \sum\frac{rs}{s-b}\cdot\frac{rs}{s-c} = \frac{r^2s^2}{(s-a)(s-b)(s-c)}\sum(s-b) =$$

$\dfrac{r^2s^3}{(s-a)(s-b)(s-c)} = s^2$ and, therefore, $\sum\dfrac{3}{2}r_b r_c = \dfrac{3s^2}{2}$.

2. Applying inequality (**1**) to $x=r_a$ and, cyclic, we obtain

$$\sum\left(\frac{r_a^2+16r_a+80}{16(r_a+4)}+\frac{2}{\sqrt{2(r_a^2+16)}}\right)bc \geq \sum\frac{3}{2}bc =$$

$$\frac{3}{2}(ab+bc+ca) = \frac{3(s^2+r^2+4Rr)}{2}.$$